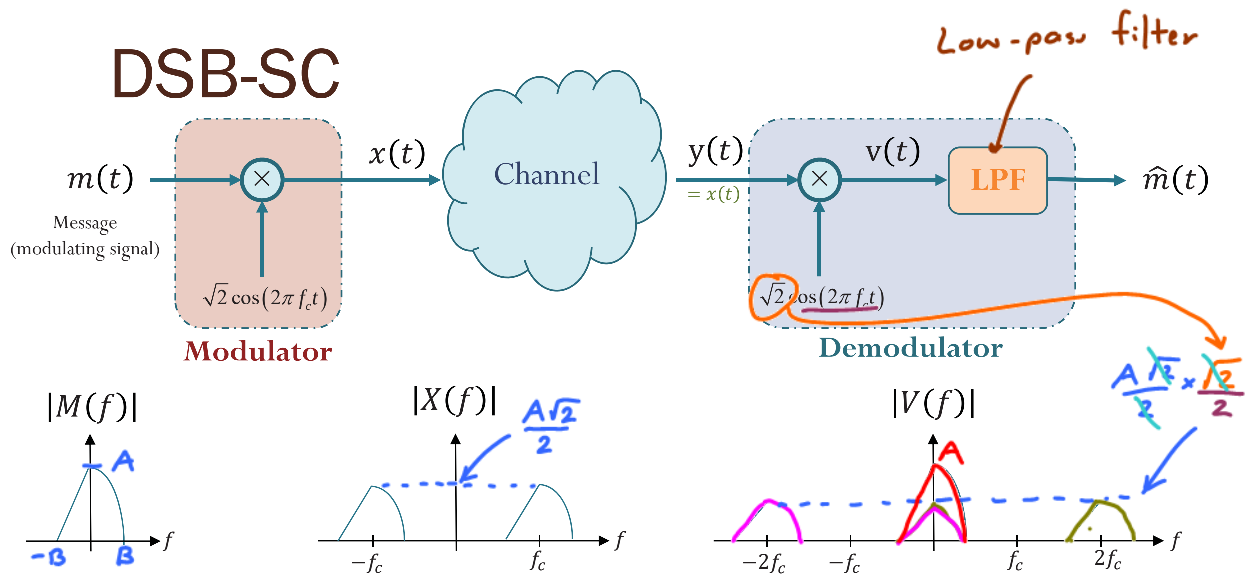
Fourier Transform and Communication Systems

Demodulation

modulation + demodulation = modem

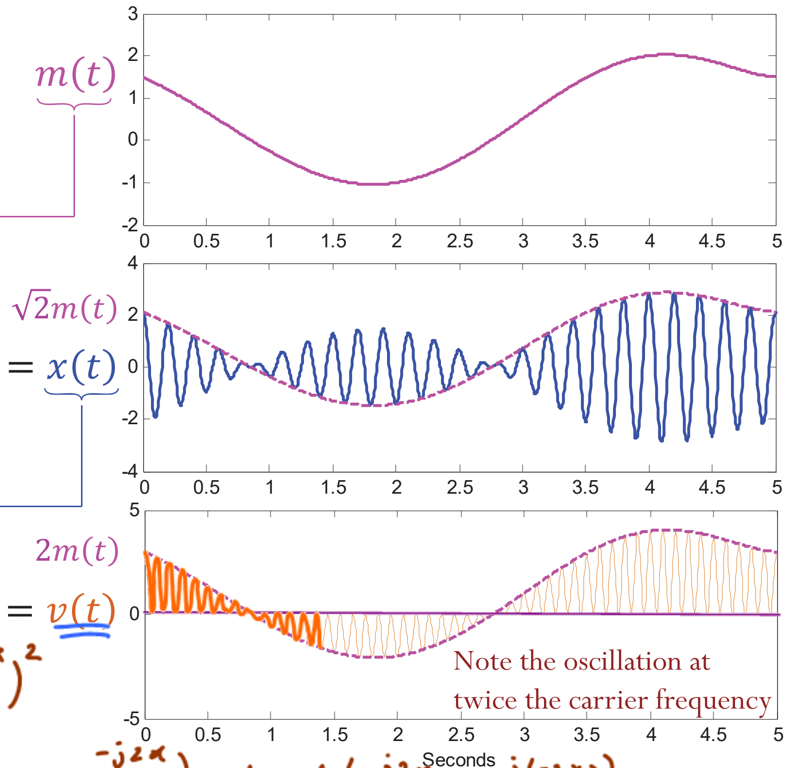
DSB-SC



Key equation:

$$\text{LPF}\left\{\underbrace{\underbrace{\left(m(t)\times\sqrt{2}\cos(2\pi f_c t)\right)}_{x(t)}\times\left(\sqrt{2}\cos(2\pi f_c t)\right)}_{v(t)}\right\} = m(t)$$

In the time domain...



$$\overbrace{m(t)}^{m(t)} \times \sqrt{2}\cos(2\pi f_c t) = \underbrace{x(t)}_{}$$

$$\overbrace{x(t)}^{m(t)\sqrt{2}\cos(2\pi f_c t)} \times \sqrt{2}\cos(2\pi f_c t) = \underline{\underline{v(t)}}$$

$$\cos^2(x) = \left(\frac{e^{jx}+e^{-jx}}{2}\right)^2$$

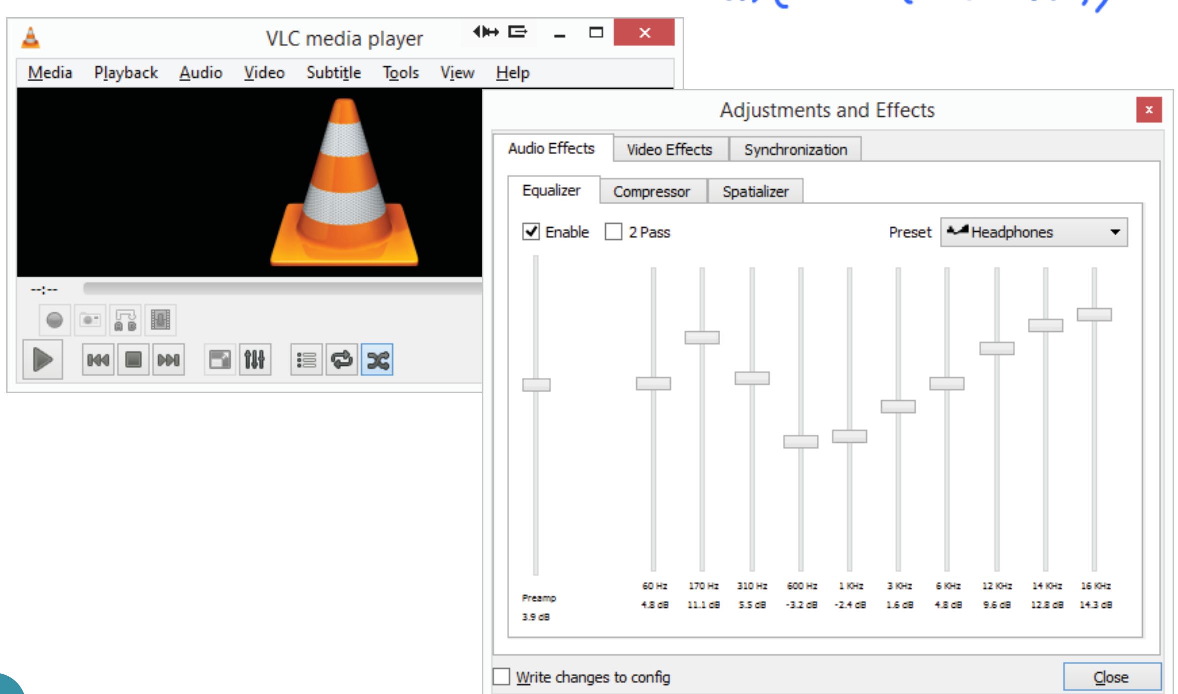
$$= \frac{1}{4}\left(e^{j2x} + 2 + e^{-j2x}\right) = \frac{1}{2} + \frac{1}{4}\left(\frac{e^{j2x}+e^{j(-2x)}}{2}\right)2$$

$$= \frac{1}{2} + \frac{1}{2}\cos(2x)$$

Scaling and Suppressing Frequency Components

$$v(t) = 2\,m(t)\cos^2(2\pi f_c t)$$

$$= m(t)\left(1 + \cos\left(2\pi(2f_c)t\right)\right)$$



Important Properties of $\mathcal{F}$:A Revisit

$\{x * y\}(t) = \int_{-\infty}^{\infty} x(\mu) y(t-\mu) d\mu = \int_{-\infty}^{\infty} x(t-\mu) y(\mu) d\mu$

Convolution Properties:

$$x * y \overset{\mathcal{F}}{\rightleftharpoons} X \times Y$$

$$x \times y \overset{\mathcal{F}}{\rightleftharpoons} X * Y$$

Shifting Properties:

$$g(t - t_0) \overset{\mathcal{F}}{\rightleftharpoons} \underbrace{e^{-j2\pi f t_0}}_{\text{Note that the magnitude of this is simply } |G(f)|} G(f)$$

$$\boxed{e^{j2\pi f_c t} g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(f - f_c)}$$

Modulation:

$$g(t)\cos(2\pi f_c t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2} G(f - f_c) + \frac{1}{2} G(f + f_c)$$

Filter Property of $\mathcal{F}$

$\{x * y\}(t) = \int_{-\infty}^{\infty} x(\mu) y(t-\mu) d\mu = \int_{-\infty}^{\infty} x(t-\mu) y(\mu) d\mu$

Convolution Properties:

$$x * y \overset{\mathcal{F}}{\rightleftharpoons} X \times Y$$

$$x \times y \overset{\mathcal{F}}{\rightleftharpoons} X * Y$$

Time Domain View:

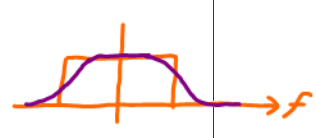
$x(t) \rightarrow$ Filter $h(t)$ $\rightarrow \{x * h\}(t)$

convolution

Frequency Domain View:

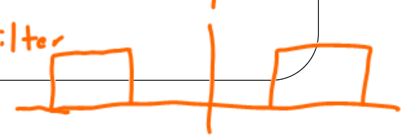
$X(f) \rightarrow$ Filter $H(f)$ $\rightarrow X(f)H(f)$

LPF : low-pass filter



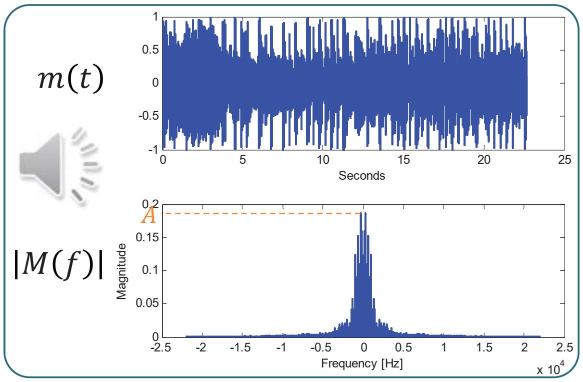
HPF : High-pass filter

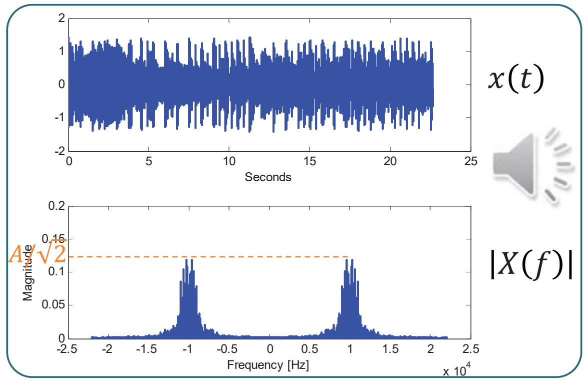
BPF : Band-pass filter

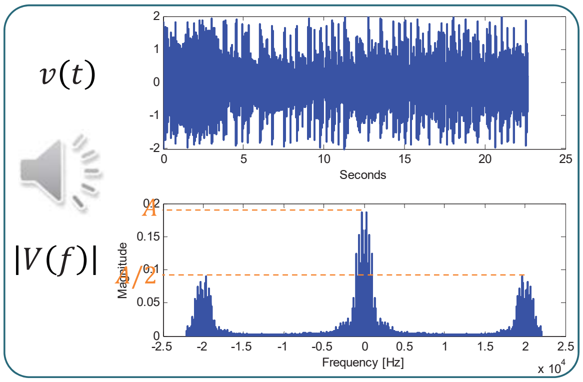


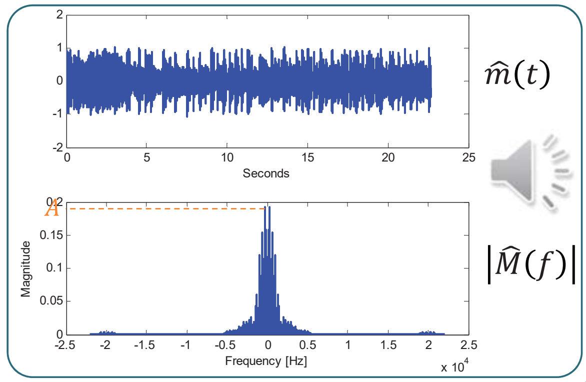
[Demo_DSBSC_Sound_ReadWAV.m]

DSB-SC



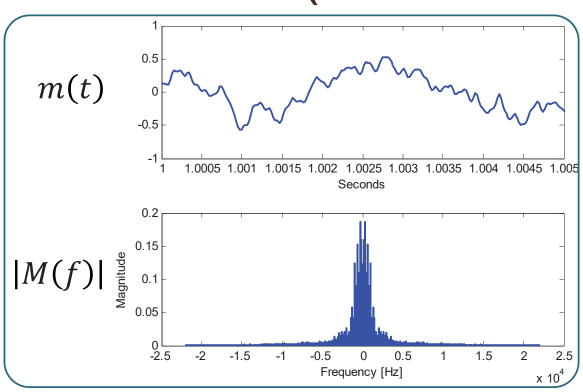


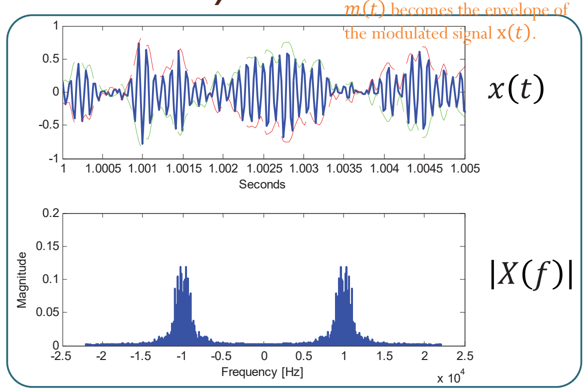


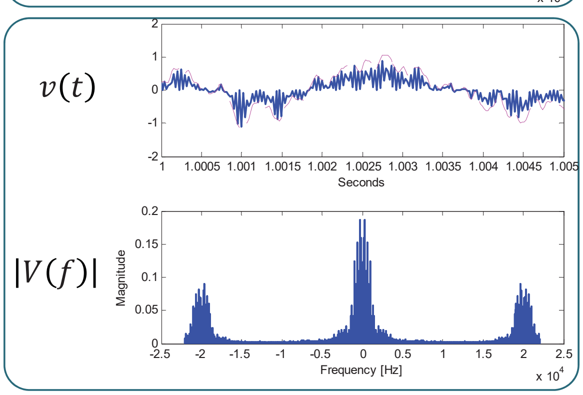


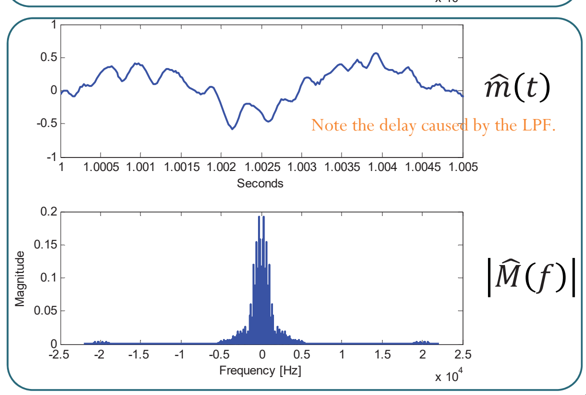
DSB-SC (Zoomed in time)

Note how the baseband signal $m(t)$ becomes the envelope of the modulated signal $x(t)$.



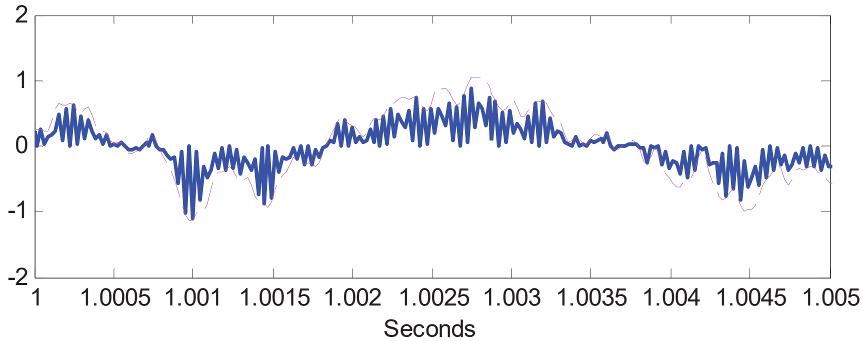






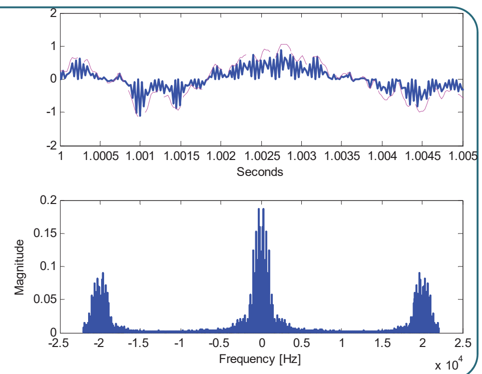
Note the delay caused by the LPF.

$v(t)$ (Zoomed in time)



$v(t)$

$|V(f)|$



In the time domain… we expect

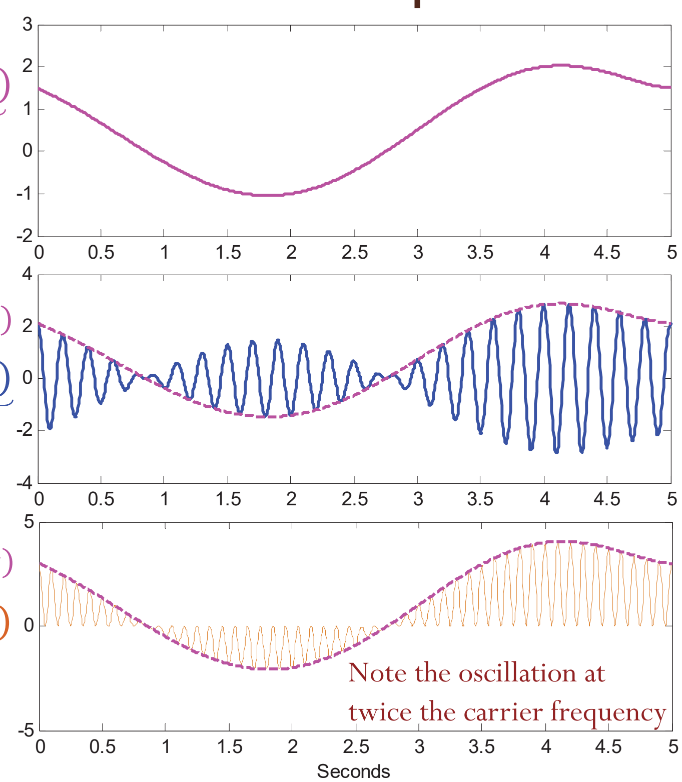
$m(t)$

$\sqrt{2}m(t)$

$$\overbrace{m(t)} \times \sqrt{2}\cos(2\pi f_c t) = \underbrace{x(t)}$$

$2m(t)$

$$\overbrace{x(t)} \times \sqrt{2}\cos(2\pi f_c t) = v(t)$$



In the time domain…

When the sampling rate is not fast enough,…

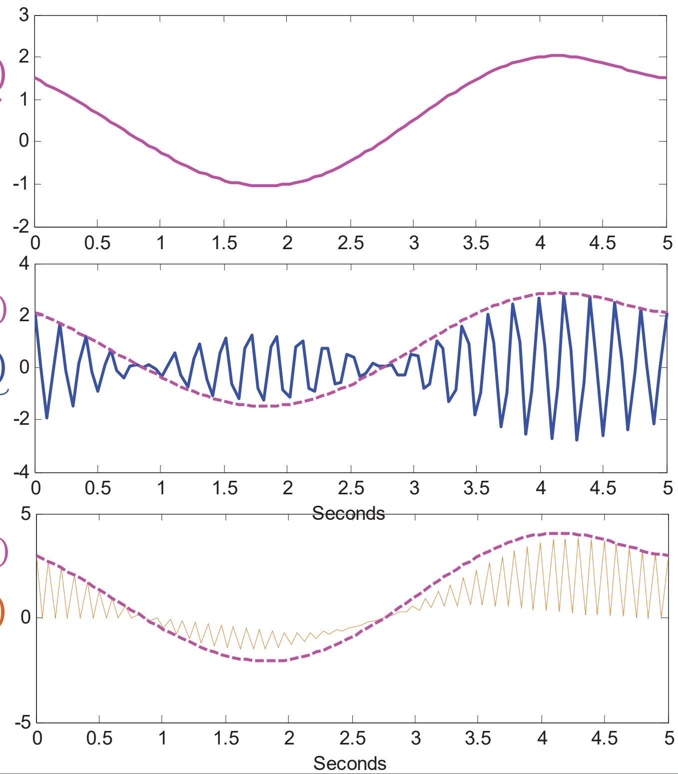
$m(t)$

$\sqrt{2}m(t)$

$$\overbrace{m(t)}\times\sqrt{2}\cos(2\pi f_c t) = \underbrace{x(t)}$$

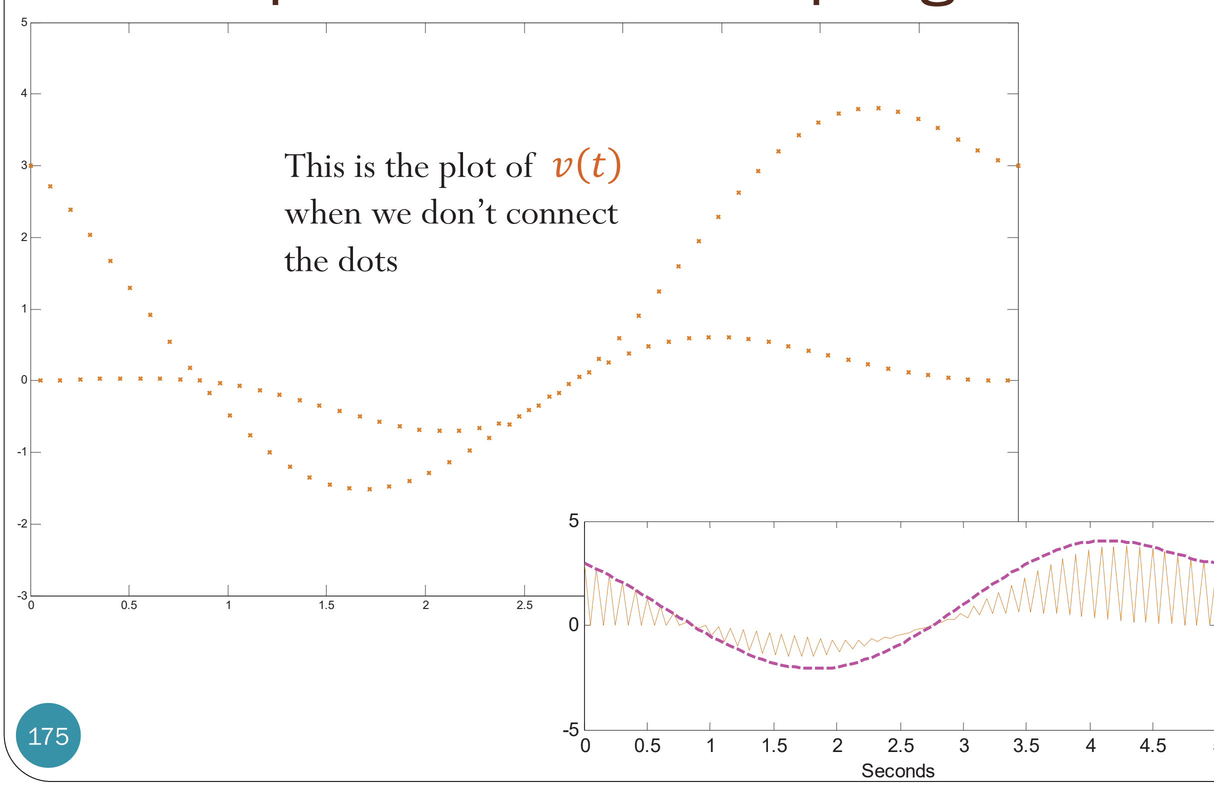
$2m(t)$

$$\overbrace{x(t)}\times\sqrt{2}\cos(2\pi f_c t) = v(t)$$

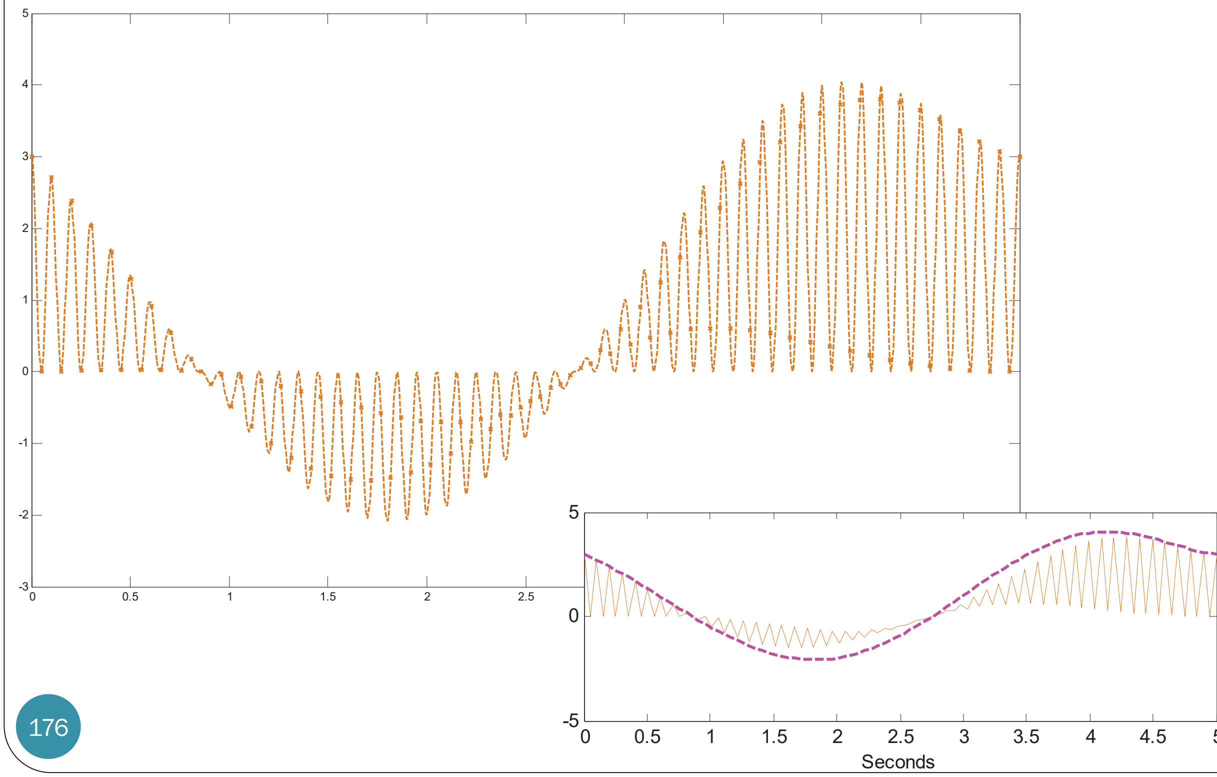


The problem with sampling rate

This is the plot of $v(t)$ when we don't connect the dots



The problem with sampling rate



BW Inefficiency in Our System (1)

Conjugate symmetry property of Fourier transform:

- Recall $M(f) = \int_{-\infty}^{\infty} m(t) e^{-j2\pi f t} dt$.
- If $m(t)$ is real-valued, then

$$M(-f) = \left(M(f)\right)^*$$

$$\int_{-\infty}^{\infty} m^*(t) e^{-(-j)2\pi f t} dt$$

$m^*(t) = m(t)$ because it is real-valued.

$$M(-f) = \int_{-\infty}^{\infty} m(t) e^{-j2\pi(-f)t} dt$$

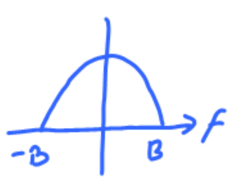
$$|M(-f)| = |M(f)|$$

Bandwidth

- The **bandwidth (BW)** of a signal is usually calculated from the differences between two frequencies (called the bandwidth limits).
- **Absolute bandwidth**: Use the highest frequency and the lowest frequency in the positive-*f* part of the signal's nonzero magnitude spectrum.
- **Half-power bandwidth** (3-dB bandwidth): Use the frequencies where the signal power starts to decrease by 3 dB (1/2).
- **Null-to-null bandwidth**: Use the signal spectrum's first set of zero crossings.
- **Occupied bandwidth**: Consider the frequency range in which X% (for example, 99%) of the energy is contained in the signal's bandwidth.

BW Inefficiency (2)

- Message bandwidth and the transmitted signal bandwidth



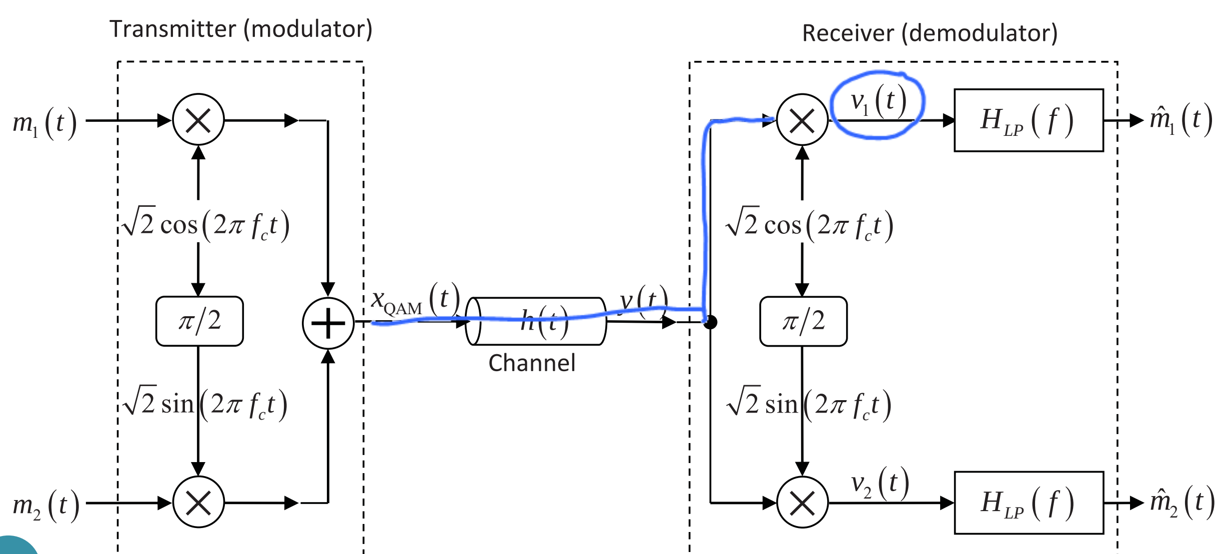
- The DSB spectrum has two sidebands: the upper sideband (USB) and the lower sideband (LSB), both containing complete information about the base-band signal $m(t)$.
- As a result, DSB signals occupy twice the bandwidth required for the baseband.

Fourier Transform and Communication Systems

Quadrature Amplitude Modulation (QAM)

QAM

- Send two messages over the same bandwidth of $2B$ Hz.
- $x_{\text{QAM}}(t) = m_1(t)\sqrt{2}\cos(2\pi f_c t) + m_2(t)\sqrt{2}\sin(2\pi f_c t)$.



A more general formula:

$$g(t)\cos(2\pi f_c t) \xrightleftharpoons{\mathcal{F}} \frac{1}{2}G(f-f_c)+\frac{1}{2}G(f+f_c).$$

$$g(t)\cos(2\pi f_c t+\phi) \xrightleftharpoons{\mathcal{F}} \frac{1}{2}\left(G(f-f_c)e^{j\phi}+G(f+f_c)e^{-j\phi}\right).$$

QAM Demodulation

- When $B < f_c$,

$$\text{LPF}\left\{x_{\text{QAM}}(t)\sqrt{2}\cos(2\pi f_c t)\right\} = m_1(t)$$

$$\text{LPF}\left\{x_{\text{QAM}}(t)\sqrt{2}\sin(2\pi f_c t)\right\} = m_2(t)$$

$$\begin{aligned}
v_1(t) &= x_{\text{QAM}}(t)\sqrt{2}\cos(2\pi f_c t)\\
&= \left(m_1(t)\sqrt{2}\cos(2\pi f_c t)+m_2(t)\sqrt{2}\sin(2\pi f_c t)\right)\sqrt{2}\cos(2\pi f_c t)\\
&= m_1(t)2\cos^2(2\pi f_c t)+m_2(t)2\sin(2\pi f_c t)\cos(2\pi f_c t)\\
&= m_1(t)\left(1+\cos(2\pi(2f_c)t)\right)+m_2(t)\sin(2\pi(2f_c)t)\\
&= m_1(t)+m_1(t)\cos(2\pi(2f_c)t)+m_2(t)\cos(2\pi(2f_c)t-90^\circ)
\end{aligned}$$

Complex form of QAM

$$x_{\text{QAM}}(t) = m_1(t)\sqrt{2}\cos(2\pi f_c t) + m_2(t)\sqrt{2}\sin(2\pi f_c t)$$
$$= \sqrt{2}\,\text{Re}\left\{m(t)e^{j2\pi f_c t}\right\},$$

$\cos(2\pi f_c t) + j\sin(2\pi f_c t)$

- where $m(t) = m_1(t) - jm_2(t)$.

$$m(t)e^{j2\pi f_c t} = m_1(t)\cos(2\pi f_c t) + jm_1(t)\sin(2\pi f_c t) - jm_2\cos(2\pi f_c t) + m_2(t)\sin(2\pi f_c t)$$

- $m(t)$: complex **envelope** or complex **baseband** signal
- $m_1(t)$: **in-phase** component
- $m_2(t)$: **quadrature**(-phase) component

QAM vs. DSB-SC Key Equations

DSB-SC Key Equation

$$\textbf{LPF}\left\{\underbrace{\left(m(t)\times\sqrt{2}\cos(2\pi f_c t)\right)}_{x_{\text{DSB-SC}}(t)}\times\left(\sqrt{2}\cos(2\pi f_c t)\right)\right\} = m(t)$$

QAM Key Equation

$$\textbf{LPF}\left\{\underbrace{\left(\text{Re}\left\{m(t)\times\sqrt{2}e^{j2\pi f_c t}\right\}\right)}_{x_{\text{QAM}}(t)}\times\left(\sqrt{2}e^{-j2\pi f_c t}\right)\right\} = m(t)$$

Derivation of the QAM Key Equation (1)

- Recall that $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$.
- Find and simplify the Fourier transform of $x^*(t)$. $\leftarrow y(t)$

$$Y(f) \equiv \int_{-\infty}^{\infty} y(t) e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} x^*(t) e^{-j2\pi f t} dt$$

$$= \left(\int_{-\infty}^{\infty} x(t) e^{-j2\pi (-f) t} dt \right)^* = \left(X(-f) \right)^*$$

$$= X^*(-f)$$

- Find and simplify the Fourier transform of $\underbrace{\mathrm{Re}\{x(t)\}}_{y(t)}$.

$$y(t) = \mathrm{Re}\{x(t)\} = \frac{1}{2}\left(x(t) + x^*(t) \right)$$

$$Y(f) = \frac{1}{2}\left(X(f) + X^*(-f) \right)$$

Derivation of the QAM Key Equation (2)

$$\mathbf{LPF}\left\{ \underbrace{\left(\mathrm{Re}\left\{ m(t) \times \sqrt{2} e^{j2\pi f_c t} \right\} \right)}_{x_{\mathrm{QAM}}(t)} \times \left(\sqrt{2} e^{-j2\pi f_c t} \right) \right\} = m(t)$$

$$m(t) \xrightarrow{\mathcal{F}} M(f)$$

$$\underbrace{m(t) e^{j2\pi f_c t}}_{g(t)} \xrightarrow{\mathcal{F}} \underbrace{M(f - f_c)}_{G(f)}$$

$$\mathrm{Re}\{g(t)\} \xrightarrow{\mathcal{F}} \frac{1}{2}\left(G(f) + G^*(-f) \right)$$

$$x_{QAM}(t) = \sqrt{2}\, \mathrm{Re}\{g(t)\}$$

$$\Big\downarrow \mathcal{F}$$

$$X_{QAM}(f) = \sqrt{2} \times \frac{1}{2}\left(G(f) + G^*(-f) \right) = \frac{1}{\sqrt{2}}\left(M(f - f_c) + M^*(-f - f_c) \right)$$

$$x_{QAM}(t)\sqrt{2}\,e^{-j2\pi f_c t} \xrightarrow{\mathcal{F}} \sqrt{2}\,X_{QAM}(f-(-f_c)) = \sqrt{2}\,X_{QAM}(f+f_c)$$

$$= M(f+\cancel{f_c}-\cancel{f_c}) + M^*(-(f+f_c)-f_c)$$

$$= M(f) + \underbrace{M^*(-(f+2f_c))}_{\text{will be eliminated by the LPF}}$$